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Nesting Directed Cycle Systems of Even Length

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It is proved that there exists a $2y$-nesting of a directed $2s$-cycle system of D_n iff $n \equiv 1 \pmod{2s}$, except possibly if $n = 4s + 1$. The existence of s-nestings of directed $2s$-cycle systems of D_n when s is odd is also considered.

1. Introduction

Let D_n denote the complete directed graph on n vertices. A *directed cycle* of a directed graph G is an ordered m-tuple $(v_0, v_1, \ldots, v_{m-1})$ such that (v_i, v_{i+1}) is an arc of G for $0 \leqslant i \leqslant m-1$ (reducing subscripts modulo m). A *directed m-cycle system* of a directed graph G is an ordered pair $(V(G), C)$, where C is a set of arc-disjoint directed cycles such that every arc of G is in exactly one directed cycle in G. A *directed (x, m)-star*, denoted by $(\{v_0, v_1, \ldots, v_{x-1}\}, \{v_x, v_{x+1}, \ldots, v_{m-1}\}, w)$ is the directed graph (V, s) with $V = \{v_0, \ldots, v_{m-1}, w\}$ and $s = \{(v_i, w) \mid 0 \leqslant i < x\} \cup \{(w, v_i) \mid x \leqslant i < m\}$. A *directed (x, m)-star system* of a directed graph G is an ordered pair $(V(G), S)$, where S is a set of arc-disjoint directed (x, m)-stars such that every arc of G is in exactly one (x, m)-star in S.

The *spectrum* of directed m-cycle systems of D_n is the set of integers n for which there exists a directed m-cycle system of D_n. Similarly, one can define the spectrum of directed m-cycle systems, and of directed (x, m)-star systems, for other families of directed graphs. It is a natural problem to find the spectrum of such structures, and indeed several results have been proved. For example, the spectrum problem for directed (x, m)-star systems has been completely settled [4], as has the spectrum problem for directed m-cycle systems of complete directed bipartite graphs [12]. The latter result is relevant to this paper, and so is formally stated here.

Theorem 1.1. *There exists a directed $2s$-cycle system of the complete directed bipartite graph $K^*_{m,n}$ iff $m \geqslant s$, $n \geqslant s$ and s divides mn.*

Similarly, the spectrum of cycle [5], path [16] and star [17] decompositions of the complete (undirected) graph K_n has also been considered, one of the outstanding problems in this area being to settle this problem for cycles. More recently, resolvable decompositions allowing paths [1], both cycles and paths [10, 11] or paths of two lengths [2] have been considered. When two graphs are allowed in the decomposition, α parallel classes of one graph and β parallel classes of the other are usually required.

In this paper, we consider another interesting variation of the mixed graph decompositions described above. We find decompositions of the complete directed graph into directed m-cycles, then another decomposition of D_n into directed (x, m)-stars, but in such a way that these two decompositions are closely related. This close relationship is defined by an x-nesting. An *x-nesting* of a directed m-cycle system of G, $(V(G), C)$, is a function f from C to the set of directed (x, m)-stars satisfying:
(1) if $f(c) = (\{v_0, \ldots, v_{x-1}\}, \{v_x, \ldots, v_{m-1}\}, w)$ *then* $V(c) = \{v_i \mid 0 \leqslant i \leqslant m-1\}$; and
(2) $\{f(c) \mid c \in C\}$ is a directed (x, m)-star system of G.

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It seems to be difficult to settle the spectrum problem of x-nested directed m-cycle systems for all x and for all m. Perhaps the most satisfying x-nestings are those with $x = 0$ or those with $x = \lfloor m/2 \rfloor$ (if $G = D_n$ then the existence of an x-nesting is equivalent to the existence of an $(m-x)$-nesting, so one may as well assume that $x \leq \lfloor m/2 \rfloor$). It has been proved that if m is odd then there exists a directed m-cycle system that has an $\lfloor m/2 \rfloor$-nesting for all $n \equiv 1 \pmod{m}$, except possibly if $n \in \{3m+1, 6m+1\}$ [9]. This essentially settles the spectrum problem for m odd and $x = \lfloor m/2 \rfloor$, since a simple counting argument proves the following result.

LEMMA 1.2. *If there exists an x-nesting of a directed m-cycle system of D_n, then $n \equiv 1$ $(\mathrm{mod}\, m)$.*

Sometimes more can be said.

LEMMA 1.3. *Let $m = 2s$. If there exists an s-nesting of a directed m-cycle system of D_{m+1}, then s is even.*

PROOF. If s is odd and $m = 2s$, then there does not exist a directed (s, m)-star system of D_{m+1} (see [4], for example). □

The spectrum problem for nestings of (undirected) m-cycle systems has also been largely settled [6–8, 13–15].

In this paper we consider $2y$-nestings and $\lfloor m/2 \rfloor$ nestings of directed m-cycle systems when m is even. From now on, we shall let $m = 2s$.

Let D_y^p denote the complete directed multipartite graph containing y vertices in each of p parts (so $D_n = D_1^n$). Let $Z_n = \{0, 1, \ldots, n-1\}$, and let $H_{2p} = \{(2i, 2i+1) \mid 0 \leq i < p\}$. Let $|y|_z$ denote the (unique) integer u such that $u \equiv y \pmod{z}$ and $-z/2 < u \leq z/2$.

2. x-NESTINGS FOR x EVEN

LEMMA 2.1 [18]. *For all positive integers $p \neq 2$, there exists a symmetric quasigroup of order $2p$ with holes of size 2.*

LEMMA 2.2. *Let $m = 2s$ and let $x = 2y \leq s$. There exists a directed m-cycle system of D_{m+1} that has an x-nesting.*

PROOF. It is well known that there exists a directed $2s$-cycle system of D_{2s+1} (one can be produced using standard difference methods). Define such a system on the vertex set Z_{2s+1}, where for $0 \leq i \leq 2s$, c_i is the directed cycle 'missing' vertex i. Define an x-nesting by $f(c_i) = (A, Z_{2s+1} \backslash (A \cup \{i\}), i)$, where $A = \{i + j \mid -y \leq j \leq y, j \neq 0\}$ (reducing all sums modulo $2s+1$). □

THEOREM 2.3. *Let $m = 2s$ and let $x = 2y \leq s$. There exists a directed m-cycle system of D_n that has an x-nesting for all $n \equiv 1$ $(\mathrm{mod}\, m)$, except possibly for $n = 2m+1$.*

PROOF. Let $n = 2ps + 1$, $p \neq 2$. Let $(Z_{2p}, \cdot)$ be a symmetric quasigroup of order $2p$ with holes of size 2 (see Lemma 2.1). Define a directed m-cycle system $(\{\infty\} \cup (Z_s \times Z_{2p}), C)$ as follows:

(1) For $0 \leq i < p$, let $(\{\infty\} \cup (Z_s \times \{2i, 2i+1\}), C_i)$ be a directed m-cycle system of D_{m+1} that has an x-nesting f_i (see Lemma 2.2). Let $C_i \subseteq C$.

(2) For $0\leq i<j<2p$, $\{i,j\}\notin H_{2p}$, let $C_{i,j}=\{c_{i,j,0},\ldots,c_{i,j,s-1}\}$ be the set of s directed m-cycles in a directed m-cycle system of D_s^2 on point set $Z_s\times\{i,j\}$ (see Theorem 1.1). Let $C_{i,j}\subseteq C$.

Clearly this defines a directed m-cycle system, so it remains to define the x-nesting f:
(a) For each cycle $c\in C_i$ define $f(c)=f_i(c)$.
(b) For $0\leq i<j<2p$, $\{i,j\}\notin H_{2p}$ and for $0\leq k\leq s-1$ define

$$f(c_{i,j,k})=(A,(Z_s\times\{i,j\})\backslash A,(k,i\cdot j)),$$

where

$$A=\begin{cases}\{(k+a,i),(k+a,j)\mid -y/2\leq a\leq y/2,\ a\neq 0\} & \text{if } y \text{ is even}\\ \{(k+a,i),(k+a,j)\mid -\lfloor y/2\rfloor\leq a\leq \lfloor y/2\rfloor\} & \text{if } y \text{ is odd.}\end{cases}$$

To see that f defines a nesting, consider the arc $((\alpha,\beta),(\gamma,\delta))$. If $\{\beta,\delta\}=\{2i,2i+1\}$, then clearly this arc is in a directed star in $f(C_i)$ (see (a)). Otherwise, suppose first that y is odd. If $-\lfloor y/2\rfloor\leq|\alpha-\gamma|_s\leq\lfloor y/2\rfloor$, then this arc is in the directed star $f(c_{\beta,z,\gamma})$, where $\beta\cdot z=\delta$, and if not then this arc is in the directed star $f(x_{\delta,z,\alpha})$, where $\delta\cdot z=\beta$. The case when y is even is analogous. □

3. x-Nestings for x Odd

In the light of Section 2, the problem of finding x-nestings when x is odd still remains. In this section, we consider possibly the most interesting special case, namely when $x=s$ and s is odd (if x is even, then x-nestings are constructed in Section 2).

The proof of Theorem 2.3 uses two ingredients: a directed m-cycle system of D_s^2 that has an x-nesting (although this was only needed to obtain directed m-cycle systems of D_{2s}^p that have an x-nesting), and a directed m-cycle system of D_{m+1}. When $x=s$ and s is odd, Lemma 1.3 proves that the vital ingredient of an s-nesting of a directed $2s$-cycle system of D_{2s+1} does not exist. However, we can circumvent this difficulty by using the following ingredients instead: an s-nesting of a directed 2s-cycle system of D_{4s+1}, and an s-nesting of a directed $2s$-cycle system of D_{4s}^p. In this section we construct these ingredients and then obtain an existence result for directed $2s$-cycle systems of D_n that have an s-nesting.

We begin by considering 1-nestings of directed 2-cycle systems of D_2^p.

Lemma 3.1. *For $p\in\{3,4,5,6,8\}$, there exists a directed 2-cycle system of D_2^p that has a 1-nesting.*

Proof. Throughout this proof, $[a,b,c]$ denotes the directed 2-cycle (a,c) for which the 1-nesting f is defined by $f((a,c))=(\{a\},\{c\},b)$. Using this notation, we consider each value of p in turn, listing the vertices in each of the p parts followed by the 1-nested directed 2-cycles.

$p=3$. The three parts are $\{\infty_0,\infty_1\}$, $\{0,2\}$ and $\{1,3\}$. Obtain the 12 1-nested directed 2-cycles by letting the permutation $(\infty_0,\infty_1)(0,1,2,3)$ act on each of $[\infty_0,0,1]$, $[0,\infty_0,1]$ and $[0,3,\infty_0]$.

$p=4$. The four parts are $\{\{(i,j),(i+2,j)\}\mid\{i,j\}\subseteq\{0,1\}\}$. The 1-nested directed 2-cycles are obtained by developing the following modulo 4:

$$[(0,0),(1,0),(0,1)]\quad [(0,0),(2,1),(1,1)]\quad [(0,1),(2,0),(1,0)]$$
$$[(0,1),(1,1),(2,0)]\quad [(0,0),(0,1),(3,0)]\quad [(0,1),(0,0),(1,1)].$$

$p=5$. The five parts are $\{\{i,5+i\}\mid 0\leq i\leq 4\}$. The 1-nested directed 2-cycles are obtained by developing the following modulo 10:

$$[0,1,8],\quad [0,3,7],\quad [0,6,4],\quad [0,2,1].$$

$p=6$. The six parts are $\{\{\infty_0, \infty_1\}\} \cup \{\{(i, 0), (i, 1)\} \mid 0 \leq i \leq 5\}$. The 1-nested directed 2-cycles are obtained by developing the following modulo 5:

$$\begin{array}{ccc} [\infty_0, (0,0), (4,0)] & [\infty_1, (0,1), (4,1)] & [(0,1), \infty_0, (1,1)] \\ [(0,1), (3,0), \infty_0] & [(0,0), (4,1), \infty_1] & [(0,0), (1,0), (3,0)] \\ [(0,0), (3,0), (1,1)] & [(0,0), (2,1), (4,1)] & [(0,1), (1,1), (3,0)] \\ & [(0,0), \infty_1, (1,0)] & \\ & [(0,1), (1,0), (2,1)] & \\ & [(0,1), (3,1), (2,0)]. & \end{array}$$

$p=8$. The eight parts are $\{\{i, 8+i\} \mid 0 \leq i \leq 7\}$. The 1-nested directed 2-cycles are obtained by developing the following modulo 16:

$$[0,7,1],\ [0,13,14],\ [0,14,13],\ [0,3,12],\ [0,5,11],\ [0,2,6],\ [0,11,7].$$ □

LEMMA 3.2. *For any integer $p \geq 3$, there exists a directed 2-cycle system of D_2^p that has a 1-nesting.*

PROOF. For all $p \geq 3$ there exists a pairwise balanced design (*PBD*) of order p having block sizes in $\{3,4,5,6,8\}$ [3, Proposition 4.2]. Apply a construction similar to [11, Lemma 3.2], using the systems constructed in Lemma 3.1. □

We shall also need the following result. A directed m-cycle system of G is *resolvable* if the directed m-cycles can be partitioned into directed 2-factors of G.

LEMMA 3.3. *There exists a resolvable directed 2s-cycle system of D_{2s}^2.*

PROOF. Let

$$c_{0,j} = ((0,0), (1,j), (0,1), (1,j+1), \ldots, (0,s-1), (1,j+s-1))$$

and

$$c_{1,j} = ((0,s), (1,s+j), (0,s+1), (1,s+j+1), \ldots, (0,2s-1), (1,2s+j-1)).$$

Then $(Z_2 \times Z_{2s}, \{c_{i,j} \mid 0 \leq i \leq 1, 0 \leq j < 2s\})$ is a resolvable directed 2s-cycle system of D_{2s}^2 with the 2s directed 2-factors being $c_{0,j}$ and $c_{1,j}$ for $0 \leq j < 2s$. □

THEOREM 3.4. *For any integer $p \geq 3$, there exists a directed 2s-cycle system of D_{4s}^p that has an s-nesting.*

PROOF. Let (Z_{2p}, C) be a directed 2-cycle system of D_2^p that has a 1-nesting f (see Lemma 3.2). The p sets of vertices in the parts are the elements of H_{2p}.

For $0 \leq k < 2s$, let $\{c_{i,j,k}, c_{i,j,k+2s}\}$ be the directed 2-factors in a resolvable directed 2s-cycle system $(Z_{2s} \times \{i, j\}, C_{i,j})$ of D_{2s}^2 (see Lemma 3.3). So $C_{i,j} = \{c_{i,j,k} \mid 0 \leq k < 4s\}$.

Define $C' = \bigcup_{0 \leq i < j < 2p, \{i,j\} \notin H_{2p}} C_{i,j}$. Then clearly $(Z_{2s} \times Z_{2p}, C')$ is a directed 2s-cycle system of D_{4s}^p in which, for $0 \leq i < p$, the set of vertices in the ith part is $Z_{2s} \times \{2i, 2i+1\}$.

For each $\{i,j\}$, $i \neq j$, $0 \leq i < 2p$, $0 \leq j < 2p$, $\{i,j\} \notin H_{2p}$, if $f((i,j)) = (\{i\}, \{j\}, \alpha)$, then for $0 \leq k < 2s$ and $z \in \{0, 2s\}$ define

$$f'(c_{i,j,k+z}) = (V(c_{i,j,k+z}) \cap (Z_{2s} \times \{i\}),\ V(c_{i,j,k+z}) \cap (Z_{2s} \times \{j\}),\ (k, \alpha)).$$

To see that f' defines an s-nesting consider the following. Since $|V(c_{i,j,k+z}) \cap (Z_{2s} \times \{i\})| = s$, it is clear that all directed stars defined by f' are directed (s,s)-stars. Consider the arc $((x,i), (k,\alpha))$, $\{i, \alpha\} \notin H_{2p}$. Since f is a 1-nesting of D_2^p, f defines a directed star containing the arc (i, α). So either $f((i, j)) = (\{i\}, \{j\}, \alpha)$ or $f((\alpha, j)) = (\{j\}, \{\alpha\}, i)$; without loss of generality we shall assume the former. Since $V(c_{i,j,k}) \cup V(c_{i,j,k+2s}) = Z_{2s} \times \{i, j\}$, either $(x, i) \in V(c_{i,j,k})$ or $(x, i) \in V(c_{i,j,k+2s})$. Hence, $(x, i) \in V(c_{i,j,k+z})$, where $z \in \{0, 2s\}$. Therefore the arc $((x, i), (k, \alpha))$ is in the directed (s, s)-star $f'(c_{i,j,k+z})$. □

LEMMA 3.5. *If $s > 1$, then there exists a directed $2s$-cycle system of D_{4s+1} that has an s-nesting.*

PROOF. It is shown in [11, Lemma 2.3] that there is an *undirected* nested $2s$-cycle system of K_{4s+1} if $s > 1$. Replace every undirected cycle by two (directed) cycles directed in opposite directions. Define a directed s-nesting of each of the two directed cycles in such a way that we use both directed edges corresponding to each undirected edge of the undirected nesting. □

THEOREM 3.6. *Let $s > 1$ be odd. Then for all $n \equiv 1 \pmod{4s}$, except possibly $n = 8s + 1$, there exists a directed $2s$-cycle system of D_n that has an s-nesting.*

PROOF. Let $n = 4sp + 1$, $p \neq 2$. Let $(Z_{2s} \times Z_{2p}, C')$ be a directed $2s$-cycle system of D^p_{4s} that has an s-nesting f' (see Theorem 3.4). Let the p parts be the elements of $Z_{2s} \times h$, $h \in H_{2p}$. For each $\{i, j\} \in H_{2p}$, $i \neq j$, let $(\{\infty\} \cup (Z_{2s} \times \{i, j\}), C_{i,j})$ be a directed $2s$-cycle system of D_{4s+1} that has an s-nesting $f_{i,j}$. Let $C = (\bigcup_{\{i,j\} \in H_{2p}} C_{i,j}) \cup C'$. Then clearly $(\{\infty\} \cup (Z_{2s} \times Z_{2p}), C)$ is a directed $2s$-cycle system of D_n that has an s-nesting f defined by

$$f(c) = \begin{cases} f'(c) & \text{if } c \in C' \\ f_{i,j}(c) & c \in C_{i,j}. \end{cases}$$ □

4. COMMENTS AND OPEN PROBLEMS

The obvious problems left to consider are to find directed m-cycle systems of D_n that have an x-nesting in the following cases:

(1) x is even and $n = 2m + 1$ (these are exceptions to Theorem 2.3);
(2) $x = s$, s is odd, ,$m = 2s$ and $n = 4m + 1$ (these are exceptions to Theorem 3.6);
(3) $x = s$, s is odd, $m = 2s$ and $n \equiv 2s + 1$ (modulo $4s$);
(4) $x < s$, s is odd and $m = 2s$.

Case (3) could be settled by using Theorem 3.4 together with the following ingredients: a solution when $x = s$ and $n = 6s + 1$; and an s-nesting of a directed $2s$-cycle system of $D_{6s+1} - D_{2s}$ (that is, D_{6s+1} wih a hole of size $2s$).

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